

# GRADIENT ESTIMATES OF MEAN CURVATURE EQUATION WITH NEUMANN BOUNDARY CONDITION IN $M^n \times \mathbb{R}$

JINJU XU AND DEKAI ZHANG

ABSTRACT. In this note, we study the prescribed mean curvature equation with Neumann boundary conditions on Riemannian product manifold  $M^n \times \mathbb{R}$ . The main goal is to establish the boundary gradient estimates for solutions by the maximum principle. As a consequence, we obtain an existence result.

Mean Curvature Equation; Neumann Boundary Condition; Gradient Estimates; Riemannian Product Manifold

## 1. INTRODUCTION

In this paper, we consider the following Neumann problem

$$(1.1) \quad \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(x, u) \quad \text{in } \Omega,$$

$$(1.2) \quad \langle Du, \gamma \rangle = \psi(x, u) \quad \text{on } \partial\Omega,$$

where  $\Omega$  is a bounded domain in  $n$ -dimensional simply connected and complete manifold  $M \subset \mathbb{R}^{n+1}$  with Riemannian metric  $\sigma$ ,  $f$  and  $\psi$  are given functions on  $M \times \mathbb{R}$  and  $\partial\Omega \times \mathbb{R}$  respectively,  $\gamma$  is the unit inner normal to  $\partial\Omega \times \mathbb{R}$ , and  $\langle, \rangle$  denotes the Riemannian metric in  $M$ .

Let  $S = \{(x, u(x)) : x \in \Omega\}$  be  $n$  dimensional graphs of mean curvature  $H$  in an  $n+1$  dimensional Riemannian manifold of the form  $M^n \times \mathbb{R}$ . If  $ds^2 = \sigma_{ij}dx_i dx_j$  is a local Riemannian metric on  $M$ , then  $M \times \mathbb{R}$  is given the product metric  $ds^2 + dt^2$  where  $t$  is a coordinate for  $\mathbb{R}$ . Then the height function  $u(x) \in C^2(\Omega)$  satisfies the following equation

$$(1.3) \quad \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = nH(x),$$

where the divergence and gradient  $Du$  are taken with respect to the metric on  $M$  i.e.

$$Du := \sum_{i=1}^n u^i \frac{\partial}{\partial x_i}, \quad u^i = \sum_{j=1}^n \sigma^{ij} \frac{\partial u}{\partial x_j}.$$

The equation (1.1) is equivalent to the following

$$(1.4) \quad \sum_{i,j=1}^n g^{ij} D_i D_j u = f(x, u)v \quad \text{in } \Omega, \quad v = \sqrt{1+|Du|^2},$$

where  $D$  denotes covariant differentiation on  $M$  and

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{1+|Du|^2}.$$

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For the following prescribed mean curvature equation with prescribed contact angle boundary value problem in  $\mathbb{R}^{n+1}$  ( $n \geq 2$ ),

$$(1.5) \quad \begin{cases} \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x, u) & \text{in } \Omega, \\ \langle Du, \gamma \rangle = \phi(x)v & \text{on } \partial\Omega, \end{cases}$$

Ural'tseva [15] first got the boundary gradient estimates and the corresponding positive gravity case existence theorem. At the same time, Simon-Spruck [11] and Gerhardt [4] also obtained existence theorem on the positive gravity case. They obtained these estimates also via test function technique. Spruck [12] used the maximum principle to obtain boundary gradient estimate in two dimension for the positive gravity capillary problems. Korevaar [7] generalized his normal variation technique and got the gradient estimates for the positive gravity case in high dimension case. Lieberman [8] got the gradient estimates using a closely related maximum principle argument to Korevaar's method on more general quasilinear elliptic equations with capillary boundary value problem in zero gravity case. Recently in [17], the first author used the maximum principle to give a new proof of gradient estimates for the mean curvature equation with oblique problem including capillary boundary value problem in zero gravity case.

In Riemannian product manifolds  $M^n \times \mathbb{R}$ , Spruck [13] proved the interior gradient estimates and existence theorems of Dirichlet problem for constant mean curvature graphs. Many researchers have also studied the capillary problem, see [9] and references therein. Recently, Ma-Xu [10] used the maximum principle to get the gradient estimate for the solutions of the mean curvature equation with Neumann boundary value and obtain an existence result. Naturally, we want to know whether we can generalize the Euclidean result to Riemannian product manifolds.

In this paper, we use Ma-Xu's technique in [10] to give the boundary gradient estimates for general mean curvature equation of graphs with Neumann boundary condition in Riemannian product manifolds  $M^n \times \mathbb{R}$ . Here our proof is more simple than that in [10].

Now we state our main result.

**Theorem 1.1.** *Let  $\Omega \subset M^n$  be a bounded domain,  $n \geq 2$ ,  $\partial\Omega \in C^3$ ,  $\gamma$  is the inward unit normal vector to  $\partial\Omega$ . Suppose  $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$  is a bounded solution of (1.1), (1.2) with  $|u| \leq M_0$  where  $M_0$  is a positive constant. Assume  $f(x, z) \in C^1(\overline{\Omega} \times [-M_0, M_0])$  and  $\psi(x, z) \in C^3(\overline{\Omega} \times [-M_0, M_0])$ , and there exist positive constants  $L_1, L_2$  such that*

$$(1.6) \quad f_z(x, z) \geq 0 \quad \text{in } \overline{\Omega} \times [-M_0, M_0],$$

$$(1.7) \quad |f(x, z)| + |D_x f(x, z)| \leq L_1 \quad \text{in } \overline{\Omega} \times [-M_0, M_0],$$

$$(1.8) \quad |\psi(x, z)|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \leq L_2.$$

*Then there exists a small positive constant  $\mu_0$  such that we have the following estimate*

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \leq \max\{M_1, M_2\},$$

where  $M_1$  is a positive constant depending only on  $n, \mu_0, M_0, L_1, \sup_{\overline{\Omega}} |Ric|$ , which is from the interior gradient estimates;  $M_2$  is a positive constant depending only on  $n, \Omega, \mu_0, M_0, L_1, L_2, \sup_{\overline{\Omega}} |Ric|$  and  $d(x) = \text{dist}(x, \partial\Omega), \Omega_{\mu_0} = \{x \in \Omega : d(x) < \mu_0\}$ .

As in [13], there is an interior gradient estimates for the mean curvature equation in  $M^n \times \mathbb{R}$ . One can also use the method of Trudinger in [14] or Xu-jia Wang's method in [16] to give the interior gradient estimates for more general mean curvature equations in  $M^n \times \mathbb{R}$ . We state it in the following.

**Remark 1.2.** If  $u \in C^3(\Omega)$  is a bounded solution for the equation (1.1) with  $|u| \leq M_0$ , and if  $f \in C^1(\overline{\Omega} \times [-M_0, M_0])$  satisfies the conditions (1.6)-(1.7), then for any subdomain  $\Omega' \subset \subset \Omega$ , we have

$$\sup_{\Omega'} |Du| \leq M_1,$$

where  $M_1$  is a positive constant depending only on  $n, M_0, \text{dist}(\Omega', \partial\Omega), L_1, \sup_{\overline{\Omega}} |Ric|$ .

From the standard bounded estimates for the prescribed mean curvature equation in Concus-Finn [1] (see also Spruck [12], Ma-Xu[10]), we can also get the following existence theorem.

**Theorem 1.3.** Let  $\Omega \subset M^n$  be a bounded domain,  $n \geq 2$ ,  $\partial\Omega \in C^3$ ,  $\gamma$  is the inward unit normal vector to  $\partial\Omega$ . If  $\psi \in C^3(\overline{\Omega})$  is a given function, then the following boundary value problem

$$(1.9) \quad \begin{cases} \text{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = u & \text{in } \Omega, \\ \langle Du, \gamma \rangle = \psi(x) & \text{on } \partial\Omega, \end{cases}$$

exists a unique solution  $u \in C^2(\overline{\Omega})$ .

The rest of the paper is organized as follows. In section 2, we first give some preliminaries. Then we shall prove the main Theorem 1.1 in section 3. As a corollary, we obtain the existence result Theorem 1.3.

## 2. PRELIMINARIES

In this section, we give some notations which are mainly from [13] and [10]. Let  $x_1, \dots, x_n$  be a system of local coordinates for  $M$  with corresponding metric  $\sigma_{ij}$ . Then the coordinate vector fields for  $S$  and the upward unit normal to  $S$  is given by

$$(2.1) \quad X_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial t},$$

and

$$(2.2) \quad N = \frac{1}{v} \left( - \sum_{1 \leq i \leq n} u^i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t} \right), \quad u^i = \sum_{1 \leq j \leq n} \sigma^{ij} u_j.$$

The induced metric on  $S$  is then

$$(2.3) \quad g_{ij} = \langle X_i, X_j \rangle = \sigma_{ij} + u_i u_j,$$

with inverse

$$(2.4) \quad g^{ij} = \sigma^{ij} - \frac{u^i u^j}{v^2}.$$

The second fundamental form  $b_{ij}$  of  $S$  is given by ( $\overline{D}$  is covariant differentiation on  $M^n \times \mathbf{R}$ )

$$\begin{aligned} b_{ij} &= \langle \overline{D}_{X_i} X_j, N \rangle \\ &= \langle D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + u_{ij} \frac{\partial}{\partial t}, N \rangle \\ &= \langle \sum_{1 \leq k \leq n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} + u_{ij} \frac{\partial}{\partial t}, N \rangle \\ &= \frac{1}{v} (-\Gamma_{ij}^k \sum_{1 \leq l \leq n} u^l \sigma_{kl} + u_{ij}). \end{aligned}$$

Hence

$$(2.5) \quad b_{ij} = \frac{1}{v} D_i D_j u$$

and the equation of prescribed mean curvature  $H(x)$  is then

$$(2.6) \quad nH(x) = \frac{1}{v} \sum_{1 \leq i, j \leq n} g^{ij} D_i D_j u,$$

where we use the notation  $v = \sqrt{1 + |Du|^2}$  for convenience.

We denote by  $\Omega$  a bounded domain in  $M^n$ ,  $n \geq 2$ ,  $\partial\Omega \in C^3$ , and set

$$d(x) = \text{dist}(x, \partial\Omega),$$

and

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}.$$

It is well known that there exists a positive constant  $\mu_1 > 0$  such that  $d(x) \in C^3(\overline{\Omega}_{\mu_1})$ ,  $|Dd| = 1$ .

As in [10], we define

$$(2.7) \quad c^{ij} = \sigma^{ij} - \gamma^i \gamma^j \quad \text{in } \Omega_{\mu_1},$$

and for a vector  $\zeta \in R^n$ , we write  $\zeta'$  for the vector with  $i$ -th component  $\sum_{1 \leq j \leq n} c^{ij} \zeta_j$ . So

$$(2.8) \quad |D'u|^2 = \sum_{1 \leq i, j \leq n} c^{ij} u_i u_j.$$

### 3. PROOF OF THEOREM 1.1

Now we begin to prove Theorem 1.1, using the technique developed by Ma-Xu [10]. Here our calculation is more simple than that in [10]. We shall choose an auxiliary function which contains  $|Du|^2$  and other lower order terms. Then we use the maximum principle for this auxiliary function in  $\overline{\Omega}_{\mu_0}$ ,  $0 < \mu_0 < \mu_1$ . At last, we get our estimates.

*Proof of Theorem 1.1.*

Setting  $w = u - \psi(x, u)d$ , we choose the following auxiliary function

$$\Phi(x) = \log |Dw|^2 e^{1+M_0+u} e^{\alpha_0 d}, \quad x \in \overline{\Omega}_{\mu_0},$$

where  $\alpha_0 = |\psi|_{C^0(\overline{\Omega} \times [-M_0, M_0])} + C_0 + 1$ ,  $C_0$  is a positive constant depending only on  $n, \Omega$ .

In order to simplify the computation, we let

$$(3.1) \quad \varphi(x) = \log \Phi(x) = \log \log |Dw|^2 + h(u) + g(d),$$

where we take

$$(3.2) \quad h(u) = 1 + M_0 + u, \quad g(d) = \alpha_0 d.$$

We assume that  $\varphi(x)$  attains its maximum at  $x_0 \in \overline{\Omega}_{\mu_0}$ , where  $0 < \mu_0 < \mu_1$  is a sufficiently small number which we shall decide it later.

Now we divide three cases to complete the proof of Theorem 1.1.

Case I. If  $\varphi(x)$  attains its maximum at  $x_0 \in \partial\Omega$ , then we shall get the bound of  $|Du|(x_0)$ .

Case II. If  $\varphi(x)$  attains its maximum at  $x_0 \in \partial\Omega_{\mu_0} \cap \Omega$ , then we shall get the estimates via the interior gradient bound in Remark 1.2.

Case III. If  $\varphi(x)$  attains its maximum at  $x_0 \in \Omega_{\mu_0}$ , then we can use the maximum principle to get the bound of  $|Du|(x_0)$ .

Now all computations will be done at the point  $x_0$ .

**Case I.** If  $\varphi(x)$  attains its maximum at  $x_0 \in \partial\Omega$ , we shall get the bound of  $|Du|(x_0)$ . We choose the normal coordinate at  $x_0$ , such that

$$\sigma_{ij}(x_0) = \delta_{ij}, \quad w_1(x_0) = |Dw|(x_0).$$

We differentiate  $\varphi$  along the normal direction.

$$(3.3) \quad \frac{\partial \varphi}{\partial \gamma} = \frac{\sum_{1 \leq i \leq n} D_i(|Dw|^2) \gamma^i}{|Dw|^2 \log |Dw|^2} + h' u_\gamma + g'.$$

Since

$$(3.4) \quad D_i w = D_i u - \psi_u D_i u d - D_{x_i} \psi d - \psi \gamma^i,$$

$$(3.5) \quad |Dw|^2 = |D'w|^2 + w_\gamma^2,$$

we have

$$(3.6) \quad w_\gamma = u_\gamma - \psi_u u_\gamma d - D_{x_i} \psi \gamma^i d - \psi = 0 \quad \text{on} \quad \partial\Omega,$$

$$(3.7) \quad D_i(|Dw|^2) = D_i(|D'w|^2) \quad \text{on} \quad \partial\Omega.$$

Applying (2.7) and (3.7), it follows that

$$(3.8) \quad \begin{aligned} \sum_{1 \leq i \leq n} D_i(|Dw|^2) \gamma^i &= \sum_{1 \leq i \leq n} D_i(|D'w|^2) \gamma^i \\ &= 2 \sum_{1 \leq i, k, l \leq n} c^{kl} D_i D_k u D_l u \gamma^i - 2 \sum_{1 \leq k, l \leq n} c^{kl} D_l u D_k \psi, \end{aligned}$$

where

$$D_k \psi = D_{x_k} \psi + \psi_u D_k u.$$

Differentiating (1.2) with respect to tangential direction, we have

$$(3.9) \quad \sum_{1 \leq k \leq n} c^{kl} D_k(u_\gamma) = \sum_{1 \leq k \leq n} c^{kl} D_k \psi.$$

It follows that

$$(3.10) \quad \sum_{1 \leq i, k \leq n} c^{kl} D_k D_i u \gamma^i = - \sum_{1 \leq i, k \leq n} c^{kl} D_i u D_k(\gamma^i) + \sum_{1 \leq k \leq n} c^{kl} D_k \psi.$$

Inserting (3.10) into (3.8) and combining (1.2), (3.3), we have

$$(3.11) \quad |Dw|^2 \log |Dw|^2 \frac{\partial \varphi}{\partial \gamma}(x_0) = (g'(0) + h' \psi) |Dw|^2 \log |Dw|^2 - 2 \sum_{1 \leq i, k, l \leq n} c^{kl} D_i u D_l u D_k(\gamma^i).$$

From (3.4), we obtain

$$(3.12) \quad |Dw|^2 = |Du|^2 - \psi^2 \quad \text{on} \quad \partial \Omega.$$

Assume  $|Du|(x_0) \geq \sqrt{100 + 2|\psi|_{C^0(\bar{\Omega} \times [-M_0, M_0])}^2}$ , otherwise we get the estimates. At  $x_0$ , we have

$$(3.13) \quad \frac{1}{2} |Du|^2 \leq |Dw|^2 \leq |Du|^2 \quad \text{and} \quad |Dw|^2 \geq 50.$$

Inserting (3.13) into (3.11), and by the choice of  $\alpha_0$ , we have

$$(3.14) \quad \begin{aligned} \frac{\partial \varphi}{\partial \gamma}(x_0) &\geq \alpha_0 - |\psi|_{C^0(\bar{\Omega} \times [-M_0, M_0])} - \frac{2 \sum_{1 \leq i, k, l \leq n} |c^{kl} D_i u D_l u D_k(\gamma^i)|}{|Dw|^2 \log |Dw|^2} \\ &\geq \alpha_0 - |\psi|_{C^0(\bar{\Omega} \times [-M_0, M_0])} - C_0 \\ &> 0. \end{aligned}$$

On the other hand, it is obvious to get

$$\frac{\partial \varphi}{\partial \gamma}(x_0) \leq 0,$$

which is a contradiction to (3.14).

Then we have

$$(3.15) \quad |Du|(x_0) \leq \sqrt{100 + 2|\psi|_{C^0(\bar{\Omega} \times [-M_0, M_0])}^2}.$$

**Case II.**  $x_0 \in \partial \Omega_{\mu_0} \cap \Omega$ . This is due to interior gradient estimates. From Remark 1.2, we have

$$(3.16) \quad \sup_{\partial \Omega_{\mu_0} \cap \Omega} |Du| \leq \tilde{M}_1,$$

where  $\tilde{M}_1$  is a positive constant depending only on  $n, M_0, \mu_0, L_1$ .

**Case III.**  $x_0 \in \Omega_{\mu_0}$ .

In this case,  $x_0$  is a critical point of  $\varphi$ . We choose the normal coordinate at  $x_0$ , such that

$$\sigma_{ij}(x_0) = \delta_{ij}, \quad w_1(x_0) = |Dw|(x_0).$$

And the matrix  $(D_i D_j w(x_0)) (2 \leq i, j \leq n)$  is diagonal. Let

$$\mu_2 \leq \frac{1}{100L_2}$$

such that

$$(3.17) \quad |\psi_u|_{\mu_2} \leq \frac{1}{100}, \quad \text{then} \quad \frac{99}{100} \leq 1 - \psi_u \mu_2 \leq \frac{101}{100}.$$

We can choose

$$\mu_0 = \frac{1}{2} \min\{\mu_1, \mu_2, 1\}.$$

In order to simplify the calculations, we let

$$w = u - G, \quad G = \psi(x, u)d.$$

Then we have

$$(3.18) \quad w_k = (1 - G_u)u_k - G_{x_k}.$$

Since at  $x_0$ ,

$$(3.19) \quad |Du|^2 = u_1^2 + \sum_{2 \leq i \leq n} u_i^2,$$

$$(3.20) \quad (1 - G_u)u_i = G_{x_i} = \psi_{x_i}d + \psi d_i, \quad i = 2, \dots, n,$$

$$(3.21) \quad w_1 = (1 - G_u)u_1 - G_{x_1} = (1 - G_u)u_1 - \psi_{x_1}d - \psi d_1.$$

So from the above relations, at  $x_0$ , we can assume

$$(3.22) \quad u_1(x_0) \geq 200(1 + |\psi|_{C^1(\overline{\Omega} \times [-M_0, M_0])}),$$

then

$$(3.23) \quad \frac{10}{11}w_1^2 \leq |Du|^2 \leq \frac{11}{9}w_1^2, \quad \frac{19}{20}u_1 \leq w_1 \leq \frac{21}{20}u_1,$$

and by the choice of  $\mu_0$  and (3.17), we have

$$(3.24) \quad \frac{99}{100} \leq 1 - G_u \leq \frac{101}{100}.$$

From the above choice, we shall prove Theorem 1.1 with two steps. As we mentioned before, all the calculations will be done at the fixed point  $x_0$ . In the following, we denote by  $D_i u = u_i$ ,  $D_j D_i u = u_{,ij}$ ,  $D_k D_j D_i u = u_{,ijk}, \dots$

**Step 1:** We first get the formula (3.29) and the estimate (3.32).

Taking the first covariant derivatives of  $\varphi$ ,

$$(3.25) \quad \varphi_i = \frac{(|Dw|^2)_i}{|Dw|^2 \log |Dw|^2} + h' u_i + g' d_i.$$

From  $\varphi_i(x_0) = 0$ , we have

$$(3.26) \quad (|Dw|^2)_i = -|Dw|^2 \log |Dw|^2 (h' u_i + g' d_i).$$

Taking the covariant derivatives again for  $\varphi_i$ , we have

$$(3.27) \quad \begin{aligned} \varphi_{,ij} = & \frac{(|Dw|^2)_{,ij}}{|Dw|^2 \log |Dw|^2} - (1 + \log |Dw|^2) \frac{(|Dw|^2)_i (|Dw|^2)_j}{(|Dw|^2 \log |Dw|^2)^2} \\ & + h' u_{,ij} + h'' u_i u_j + g'' d_i d_j + g' d_{,ij}. \end{aligned}$$

Using (3.26), it follows that

$$(3.28) \quad \begin{aligned} \varphi_{,ij} = & \frac{(|Dw|^2)_{,ij}}{|Dw|^2 \log |Dw|^2} + h' u_{,ij} + [h'' - (1 + \log |Dw|^2) h'^2] u_i u_j + g' d_{,ij} \\ & + [g'' - (1 + \log |Dw|^2) g'^2] d_i d_j - (1 + \log |Dw|^2) h' g' (d_i u_j + d_j u_i). \end{aligned}$$

Then we get

$$(3.29) \quad 0 \geq \sum_{1 \leq i, j \leq n} g^{ij} \varphi_{,ij}(x_0) =: I_1 + I_2,$$

where

$$(3.30) \quad I_1 = \frac{1}{|Dw|^2 \log |Dw|^2} \sum_{1 \leq i, j \leq n} g^{ij} (|Dw|^2)_{,ij},$$

and

$$(3.31) \quad \begin{aligned} I_2 = & \sum_{1 \leq i, j \leq n} g^{ij} \{ h' u_{,ij} + [h'' - (1 + \log |Dw|^2) h'^2] u_i u_j \\ & + [g'' - (1 + \log |Dw|^2) g'^2] d_i d_j - 2(1 + \log |Dw|^2) h' g' d_i u_j + g' d_{,ij} \}. \end{aligned}$$

From the choice of the coordinate and the equations (1.4), (3.2), we have the estimate for  $I_2$ :

$$(3.32) \quad I_2 \geq f v - 2 \log w_1 \sum_{1 \leq i, j \leq n} g^{ij} u_i u_j - 2 \alpha_0^2 \log w_1 \sum_{1 \leq i, j \leq n} g^{ij} d_i d_j - C_1,$$

here  $C_1$  depending only on  $n, \Omega, M_0, \mu_0, L_2$ .

**Step 2:** We treat  $I_1$  and finish the proof of Theorem 1.1.

Taking the first covariant derivatives of  $|Dw|^2$ , we have

$$(3.33) \quad (|Dw|^2)_i = 2w_1 w_{,1i}.$$

Taking the covariant derivatives of  $|Dw|^2$  once more, we have

$$(3.34) \quad (|Dw|^2)_{,ij} = 2w_1 w_{,1ij} + 2 \sum_{1 \leq k \leq n} w_{,ki} w_{,kj}.$$

By (3.30) and (3.34), we can rewrite  $I_1$  as

$$(3.35) \quad \begin{aligned} I_1 = & \frac{1}{w_1^2 \log w_1} \left[ \sum_{1 \leq i, j \leq n} g^{ij} w_1 w_{,1ij} + \sum_{1 \leq i, j, k \leq n} g^{ij} w_{,ki} w_{,kj} \right] \\ = & \frac{1}{w_1^2 \log w_1} [I_{11} + I_{12}]. \end{aligned}$$

In the following, we shall deal with  $I_{11}$  and  $I_{12}$  respectively.

For the term  $I_{11} = w_1 \sum_{1 \leq i, j \leq n} g^{ij} w_{,1ij}$ : as we have let

$$(3.36) \quad w = u - G, \quad G = \psi(x, u)d,$$



then we have

$$\begin{aligned}
(3.37) \quad w_1 &= (1 - G_u)u_1 - G_{x_1}, \\
w_{,1i} &= (1 - G_u)u_{,1i} - G_{uu}u_1u_i - G_{ux_i}u_1 - G_{x_1u}u_i - G_{x_1x_i}, \\
w_{,1ij} &= (1 - G_u)u_{,1ij} - G_{uu}(u_{,1i}u_j + u_{,1j}u_i + u_{,ij}u_1) \\
&\quad - G_{ux_i}u_{,1j} - G_{ux_j}u_{,1i} - G_{x_1u}u_{,ij} - G_{uuu}u_1u_iu_j \\
&\quad - G_{uux_j}u_1u_i - G_{ux_iu}u_1u_j - G_{x_1uu}u_iu_j \\
(3.38) \quad &\quad - G_{ux_ix_j}u_1 - G_{x_1ux_j}u_i - G_{x_1x_iu}u_j - G_{x_1x_ix_j}.
\end{aligned}$$

So from the choice of the coordinate and the equations (1.4), (3.38), we have

$$\begin{aligned}
(3.39) \quad \sum_{1 \leq i, j \leq n} g^{ij}w_{,1ij} &\geq (1 - G_u) \sum_{1 \leq i, j \leq n} g^{ij}u_{,1ij} - 2 \sum_{1 \leq i, j \leq n} g^{ij}(G_{uu}u_i + G_{ux_i})u_{,1j} \\
&\quad - fG_{uu}vu_1 - C_2u_1.
\end{aligned}$$

Differentiating (1.4), we have

$$(3.40) \quad g^{ij}u_{,ij1} = - \sum_{1 \leq l \leq n} g_{pl}^{ij}u_{,l1}u_{,ij} + vD_1f + fv_1.$$

From (2.4), we have

$$(3.41) \quad g_{pl}^{ij} = - \frac{1}{v^2}(\delta_{il}u_j + \delta_{jl}u_i) + \frac{2}{v^4}u_iu_ju_l.$$

By the definition of  $v$ , we have

$$(3.42) \quad vv_1 = \sum_{1 \leq l \leq n} u_lu_{,l1}.$$

and

$$(3.43) \quad D_1f = f_uu_1 + f_{x_1}.$$

Inserting (3.41), (3.42) and (3.43) into (3.40), we have

$$(3.44) \quad g^{ij}u_{,ij1} = \frac{2}{v^2} \sum_{1 \leq l \leq n} g^{il}u_{,j}u_{,ij}u_{,l1} + f \sum_{1 \leq l \leq n} \frac{u_lu_{,l1}}{v} + f_uvu_1 + f_{x_1}v.$$

From the Ricci identity, we have

$$(3.45) \quad u_{,ij1} = u_{,1ij} + \sum_{1 \leq l \leq n} u_lR_{ij1}^l.$$

Inserting (3.45) into (3.44), we have

$$(3.46) \quad g^{ij}u_{,1ij} = \frac{2}{v^2} \sum_{1 \leq l \leq n} g^{il}u_{,j}u_{,ij}u_{,l1} - \sum_{1 \leq l \leq n} u_lR_{ij1}^l + f \sum_{1 \leq l \leq n} \frac{u_lu_{,l1}}{v} + f_uvu_1 + f_{x_1}v.$$

Inserting (3.46) into (3.39), we have the formula for  $I_{11}$

$$\begin{aligned}
(3.47) \quad I_{11} &\geq \frac{2(1 - G_u)w_1}{v^2} \sum_{1 \leq i, j, l \leq n} g^{il}u_{,j}u_{,ij}u_{,l1} - 2(1 - G_u)w_1 \sum_{1 \leq i, j \leq n} g^{ij}(G_{uu}u_i + G_{ux_i})u_{,1j} \\
&\quad + f(1 - G_u)w_1 \sum_{1 \leq l \leq n} \frac{u_lu_{,l1}}{v} - fG_{uu}(1 - G_u)w_1vu_1 - C_3u_1^2.
\end{aligned}$$

In the above inequality, we have used  $f_u \geq 0$ .

For the term  $I_{12}$ :

$$(3.48) \quad I_{12} \geq \sum_{1 \leq i, j \leq n} g^{ij} w_{1i} w_{1j} + \sum_{2 \leq i \leq n} g^{ii} w_{ii}^2.$$

In the following we use the relation  $\varphi_i(x_0) = 0$ , we get the formula

$$(3.49) \quad w_{1i} = -w_1 \log w_1 (h' u_i + g' d_i).$$

From (3.37) and (3.49), we have

$$(3.50) \quad (1 - G_u) u_{,1i} = -w_1 \log w_1 (h' u_i + g' d_i) + [G_{uu} u_1 u_i + G_{ux_i} u_1 + G_{x_1 u} u_i + G_{x_1 x_i}].$$

So putting (3.37), (3.49), (3.50) into (3.47) and (3.48), we have

$$(3.51) \quad \begin{aligned} I_{11} + I_{12} &\geq \frac{2(1 - G_u)w_1}{v^2} \sum_{1 \leq i, j, l \leq n} g^{il} u_{,j} u_{,ij} u_{,l1} + \sum_{1 \leq i, j \leq n} g^{ij} w_{1i} w_{1j} + \sum_{2 \leq i \leq n} g^{ii} w_{ii}^2 \\ &\quad - 2(1 - G_u)w_1 \sum_{1 \leq i, j \leq n} g^{ij} (G_{uu} u_i + G_{ux_i}) u_{,1j} + f(1 - G_u)w_1 \sum_{1 \leq l \leq n} \frac{u_l u_{,l1}}{v} \\ &\quad - f G_{uu} (1 - G_u) w_1 v u_1 - C_3 u_1^2 \\ &\geq 3h'^2 w_1^2 \log^2 w_1 \sum_{1 \leq i, j \leq n} g^{ij} u_i u_j + 3g'^2 w_1^2 \log^2 w_1 \sum_{1 \leq i, j \leq n} g^{ij} d_i d_j \\ &\quad - h' f w_1^2 \log^2 w_1 \frac{u_1^2}{v} - C_4 w_1^2 \log w_1 \\ &= 3w_1^2 \log^2 w_1 \sum_{1 \leq i, j \leq n} g^{ij} u_i u_j + 3\alpha_0^2 w_1^2 \log^2 w_1 \sum_{1 \leq i, j \leq n} g^{ij} d_i d_j \\ &\quad - f w_1^2 \log^2 w_1 \frac{u_1^2}{v} - C_4 w_1^2 \log w_1, \end{aligned}$$

The above last equality has been used the choice of  $h$  and  $g$ . It follows that from (3.35),

$$(3.52) \quad I_1 \geq 3 \log w_1 \sum_{1 \leq i, j \leq n} g^{ij} u_i u_j + 3\alpha_0^2 \log w_1 \sum_{1 \leq i, j \leq n} g^{ij} d_i d_j - f \log w_1 \frac{u_1^2}{v} - C_4.$$

Combining (3.52), (3.32) and (3.29), it follows that

$$(3.53) \quad 0 \geq \sum_{1 \leq i, j \leq n} g^{ij} \varphi_{,ij}(x_0) \geq \frac{1}{4} \log w_1 - C_5.$$

So there exists a positive constant  $C_6$  such that

$$(3.54) \quad |Du|(x_0) \leq C_6.$$

So from Case I, Case II, and (3.54), we have

$$|Du|(x_0) \leq C_7, \quad x_0 \in \Omega_{\mu_0} \bigcup \partial\Omega.$$

Here the above  $C_2, \dots, C_7$  are positive constants depending only on  $n, \Omega, \mu_0, M_0, \sup_{\overline{\Omega}} |Ric|, L_1, L_2$ .

Since  $\varphi(x) \leq \varphi(x_0)$ , for  $x \in \Omega_{\mu_0}$ , there exists  $M_2$  such that

$$(3.55) \quad |Du|(x) \leq M_2, \quad \text{in } \Omega_{\mu_0} \bigcup \partial\Omega,$$

where  $M_2$  depends only on  $n, \Omega, \sup_{\overline{\Omega}} |Ric|, \mu_0, M_0, \sup_{\overline{\Omega}} |Ric|, L_1, L_2$ .

So at last we get the following estimate

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \leq \max\{M_1, M_2\},$$

where the positive constant  $M_1$  depends only on  $n, \mu_0, M_0, \sup_{\overline{\Omega}} |Ric|, L_1$ ; and  $M_2$  depends only on  $n, \Omega, \mu_0, M_0, \sup_{\overline{\Omega}} |Ric|, L_1, L_2$ .

So we complete the proof of Theorem 1.1.  $\square$

As a consequence, from the standard  $C^0$  estimates for the prescribed mean curvature equation in Concus-Finn [1] ( see also Spruck [12], Ma-Xu[10]), we can also get the existence theorem Theorem 1.3.

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DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444 CHINA  
E-mail address: jjxujane@shu.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI ANHUI 230026 CHINA  
E-mail address: dekzhang@mail.ustc.edu.cn